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PROGRESS REPORT.

Mr. V. Anantharaman, Student, 60 Sarvamanya Street, Melakavari, has been elected a member at concessional rate.

2. Under Art. VIII (*c*) of the Constitution, the President takes this opportunity to announce that the Committee have re-elected Messrs. M. T. Naraniengar and D. D. Kapadia as Honorary Joint Secretaries for the ensuing two years.

3. Under instructions from the Committee, I have the pleasure to inform the General Body that Mr. P. R. Krishnaswami, M.A., Assistant Professor of Mathematics, Pachaiyappa's College, Madras has been appointed Assistant Secretary of the Society in place of Mr. P. V. Seshu Aiyar, B.A., L.T., Professor of Mathematical Physics, Presidency College, Madras who takes his seat on the Committee. The best thanks of the Committee are due to Mr. Seshu Aiyar for his able and zealous work as Assistant Secretary for the last five years. The Committee trusts that his successor will prove equally useful to the Society.

4. The Committee regret to record the untimely death of Rao Saheb B. J. Naik, M.A., Professor of Mathematics, D. J. Sind College, Karachi—an able and unassuming member of our Society.

5. The following books have been received for review :—

1. *Calculus made Easy*—by F.R.S. (second edition), Macmillan & Co., London, 1914, 2*s. net*;

2. *Workshop Arithmetic*—by F. Castle, Macmillan & Co., London, 1915, 1*s.* 6*d.*;

3. *Plane Trigonometry*—by H. S. Carslaw (corrected edition), Macmillan & Co., London, 1915, 4*s*. 6*d*.;

4. *The Laws of Algebra*—by A. G. Craknell, Univ. Tut. Press, London, 1915, 1*s*.

5. *Junior Algebra*—by A. G. Cracknell & A. Barraclough, Univ. Tut. Press, London, 1915, 2*s*.

6. According to Art. VI of the Bye-laws the audited Balance Sheet for the year 1914, and the Budget for the current year are circulated. In doing so, the Committee cannot but notice with a sense of apprehension the *decrease* in the receipt of subscriptions. It is confidently expected that members will try their utmost to pay off arrears and current subscriptions.

POONA,
31st March 1915.

D. D. KAPADIA,
Hony. Joint Secretary.

Balance Sheet for 1914.

Receipts.	Rs.	A.	P.	*Expenditure.*	Rs.	A.	P.
Opening Balance from 1913 ...	970	1	9	Books and Journals ...	287	4	9
Subscriptions from Members ...	629	0	0	Library ...	309	4	0
„ for Journal.	122	13	0	Journal Printing ...	433	1	6
Miscellaneous ...	0	14	0	Ordinary Working Expenses ...	184	8	4
				Closing Balance ...	508	10	2
	1,722	12	9		1,722	12	9

MADRAS,
12th *February* 1915.

(Sd.) S. NARAYANA AIYAR,
Ag. Hon. Treasurer.

I have examined the Treasurer's books and vouchers and the monthly statements submitted by the Secretary, Assistant Secretary and Assistant Librarian and declare the above accounts to be correct.

MADRAS,
12th *February* 1915.

(Sd.) S. KRISHNASWAMI AIYAR,
Auditor.

Budget for 1915.

Receipts.	Rs.	A.	P.	*Expenditure.*	Rs.	A.	P.
Opening Balance ...	508	10	2	Books and Journals ...	350	0	0
Subscriptions from Members ...	2,100	0	0	Library ...	350	0	0
Sale of Journals ...	150	0	0	Printing Journal ...	550	0	0
				Ordinary Working Expenses ...	250	0	0
				Closing Balance ...	1,258	10	2
	2,758	10	2		2,758	10	2

MADRAS,
9th *February* 1915.

(Sd.) S. NARAYANA AIYAR,
Ag. Hon. Treasurer.

So-called Cases of Failure in the Solution of Linear Differential Equations.

By Eric H. Neville.

(*Continued from page* 6.)

We turn now to the question of the generality of the solution of equation (2) obtainable by means of equation (3), that is, to the problem of the construction of integrals of the equation

$$G(y)=0 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (17)$$

In passing we remark that it is shewn by the last example that integrals of equation (2) deduced from different functions satisfying the conditions of equation (3) may differ, so that a number of independent functions satisfying equations of the form of (3) may lead to an equivalent number of independent solutions of (2), or may lead to any smaller number other than zero.

Again, any arbitrary constants occurring in the function $F(x, p_1, p_2, \ldots p_r)$ may be regarded as arbitrary functions of any or all of the parameters $p_1, p_2, \ldots p_r$ without affecting the relation between the equations (2), (3), the only conditions which may have to be imposed being negative conditions, preventing them or certain combinations of them from becoming zero or infinite for the special values $a_1, a_2, \ldots a_r$ of the parameters; it is easy to prove, from the way in which the arbitrary constants must enter into the function F, that the derived solution of equation (2) contains the same number of arbitrary constants as the given function.

Considerations of a different kind appear when we relate the solution of equation (17), or in other words the discovery of terms in the complementary function of equation (2), to the given result (3).

The first conclusion from the equation (3) in this connection is obvious: if for any set of values $b_{t1}, b_{t2}, \ldots b_{tr}$ of $p_1, p_2, \ldots p_r$ the function $h(p_1, p_2, \ldots p_r)$ is zero, the function $F(x, b_{t1}, b_{t2}, \ldots b_{tr})$ is a solution of equation (17). We may call such a set a zero-set of the function h. If there is only one parameter p, the values of p for which $h(p)$ vanishes are fixed numbers, the roots of the equation $h(p)=0$, but if there are more parameters than one, we can, as a rule, fix arbitrarily all but one of the parameters and determine the remaining one by the condition that the function vanishes; this remaining parameter is a function of the parameters which we consider arbitrary, and if one of the latter varies, the dependent parameter varies also. In fact, if there are more parameters than one, the values forming a zero-set are

not determinate but are capable of continuous variation; for this reason the case of a single parameter is much the easier to discuss, and we commence with it.

The problem now before us is this: given a homogeneous linear differential operator G involving the independent variable x in any way whatever, and a function $F(x, p)$ involving a parameter p and known to be such that

$$G\{F(x, p)\} = h(p) \cdot f(x, p), \quad \ldots \quad \ldots \quad (18)$$

to construct solutions of the differential equation

$$G(y) = 0.$$

Without loss of generality we may suppose that $f(x, p)$ does not vanish or become infinite irrespectively of x for any finite value of p; on account of the form G we could make the same assumption as to $F(x, p)$, but it is in fact inconvenient to assume that $F(x, p)$ cannot vanish identically for certain values of p; so, with regard to $F(x, p)$ we assume only that it does not become identically infinite for any finite value of p.

From equation (18) and the restrictions imposed on $f(x, p)$ it follows that the necessary and sufficient condition for $F(x, p)$ to be a solution of $G(y) = 0$ is that p should be a root of the equation $h(p) = 0$; or, in other words, a zero of the function $h(p)$. Every zero of $h(p)$ will give rise to a solution of the equation, but this solution may be only the trivial one $y = 0$; the zeros of $h(p)$ which lead to integrals other than $y = 0$ may be distinguished as efficient zeros.

Again, from (18) we have, for all values of p, the equation

$$G\{\partial F/\partial p\} = (dh/dp)f(x, p) + h(p)(\partial f/\partial p), \quad \ldots \quad (19)$$

which is only the particular case, for one parameter, of an equation already written out and used. But, whereas before, we were anxious that this equation should *not* reduce to the form $G(y) = 0$, now it is precisely of this last equation that we are trying to find solutions. Even with the conditions already imposed $\partial f/\partial p$ may vanish identically for a particular value of p which is also a zero of dh/dp, but such a case is quite special. And the only other case in which the right hand side of equation (19) can be seen from its form to vanish is that in which p has a value causing both $h(p)$ and dh/dp to vanish; or, in other words, that in which $h(p)$ has a multiple zero. It is worth while to notice that if b is a simple zero of $h(p)$, the function $G\{\partial F/\partial b\}$ cannot under any circumstances vanish identically. We may differentiate equation (19) with respect to p and argue in the same way from the identity which results that $G\{\partial^2 F/\partial p^2\}$ certainly vanishes when the value

of p is a triple zero of $h(p)$ and certainly does not vanish when the value of p is a double but not a triple zero of this function, and that the only other values of p which the general form of the identity shews to furnish solutions of equation (17) are values which are simultaneously double zeros of $h(p)$ and single zeros, for all values of x, of $\partial f/\partial p$, and values which are simultaneously single zeros of $h(p)$ and double zeros, identically with respect to x, of $\partial f/\partial p$. And, in general, if $h(p)$ has zeros $b_1, b_2,\ldots$ of multiplicities $q_1, q_2,\ldots$, as many of the $\Sigma(q_k)$ functions

$$F(x, b_1),\ \partial F/\partial b_1, \ldots \partial^{q_1-1}F/\partial b_1^{q_1-1};$$
$$F(x, b_2),\ \partial F/\partial b_2, \ldots \partial^{q_2-1}F/\partial b_2^{q_2-1}; \ldots\ldots\ldots\ldots,$$

as are not identically zero are proper solutions of the differential equation $G(y)=0$.

For example, from the equation (4) it follows that the values which the functions e^{px}, $\partial e^{px}/\partial p$ assume when $p=1$, that is, the functions e^x, xe^x, are solutions of the equation

$$(D-1)^2\, y=0,$$

and incidentally it follows that the function x^2e^x is not a solution of the equation.

The functions $f(x, p)$, $F(x, p)$ are just as likely to tend to finite forms as P tends to infinity as they are to take finite forms when p takes a finite value, and if we make the general assumption that neither the function $f(x, p)$, nor the function $F(x, p)$, nor any one of the derivatives of the latter function with respect to p, becomes indentically infinite as p takes any constant value, finite or infinite, we can assert that the number of solutions of the equation $G(y)=0$ which a knowledge of a number of the zeros of the function $h(p)$ enables us to obtain, is equal to the order of $h(p)$ in the known zeros, that is to the sum of the multiplicities of these zeros; and if the knowledge of the zeros of $h(p)$ is complete, a number of solutions equal to the total order of this function can be found.

To see the importance of this result, we may consider the operator G as obtained from an operator G_u involving a number s, not necessarily greater than one, of arbitrary parameters $u_1, u_2 \ldots u_s$ by giving special values $c_1, c_2, \ldots c_s$ to these parameters, and we may suppose that a result of the form

$$G_u\{F_u(x, p)\} = h_u(p) f_u(x, p)$$

is known, the various functions involving, in addition to the variables x, p, the parameters $u_1, u_2, \ldots u_s$, and that these functions reduce to the functions we have been considering when these parameters take

the values $c_1, c_2, \ldots c_s$. Then although a variation of these parameters will usually change the positions of the zeros of the function $h_u(p)$, causing coalescence and separation, it will not as a rule cause any change in the total order of this function, and when the order of the function is unchanged so also is its value for the determination of solutions of the differential equation $G_u(y)=0$.

For example, although the particular solutions e^{lx}, e^{mx} of the equation

$$(D-l)(D-m)y=0$$

which are suggested by the result

$$(D-l)(D-m)e^{px}=\{(p-l)(p-m)\}e^{px}$$

coalesce when l and m are equal, the corresponding identity

$$(D-l)^2e^{px}=(p-l)^2e^{px}$$

still has a quadratic function of p for a factor in the right hand side, and is still effective for suggesting two solutions of the equation

$$(D-l)^2 y=0,$$

namely e^{lx} and xe^{lx}.

But it is to be observed that neither in this particular case nor in general is it necessary or even useful to have acquaintance with the solution of a more general equation under which a proposed equation is included. Unless the general equation happens to be such as to suggest to us a function $F(x, p)$ leading to a function $h(p)$ radically different from any such function already known, it is unlikely that by attacking the general equation we shall make any advance in solving the particular one.

The best illustration of this part of the theory is to be found in the method devised by Frobenius for constructing regular integrals of homogeneous linear equations. A function $F(x, p)$ is formed such that

$$G\{F(x, p)\}=h(p)\,x^p,$$

this function not becoming infinite identically for any finite value of p and $h(p)$ being a polynomial in p; the function $F(x, p)$ is constructed as a power series in x whose co-efficients are rational functions of p, and it is not always convenient to prevent these co-efficients from having a common factor in the numerator, though they must not have one in the denominator; for some of the zeros of $h(p)$ the function $F(x, p)$ may vanish identically, and it has been proved by Fuchs and by Frobenius (for references, a proof, and examples, see Forsyth's *Theory of Differential Equations*, Vol. IV) that the remaining zeros, which are the efficient zeros of the polynomial, are always sufficiently numerous to lead to all the regular integrals of the equation $G(y)=0$.

The precise result is that if the efficient zeros are divided into groups so that the real part of each zero differs from the real part of every zero in its own group by an integer and from the real part of every zero in every other group by an amount which is not an exact integer, and if the members of each group are so arranged that the real part of each is greater than the real part of the zero which follows it in its own group, then multiplicity q_{kl} of b_{kl}, the lth member of the kth group, is greater than the multiplicity $q_{k,\,l-1}$ of the zero $b_{k,\,l-1}$ which precedes it in its own group, and if we denote the value of $\partial^m F/\partial p^m$ when p has the value b_{kl} by $F_{kl}(m)$, the set of functions obtained by forming from each zero b_{kl} the $q_{kl}-q_{k,\,l-1}$ functions

$$F_{kl}(q_{k,\,l-1}),\ F_{kl}\,(q_{k,\,l-1}+1),\ F_{kl}(q_{k,\,l-1}+2),\ldots\ldots F_{kl}\,(q_{kl}-1)$$

is a complete set of linearly independent regular integrals of the equation $G(y)=0$ (the functions associated with the first member b_{k1} of the group are the values of F itself and of its first $q_{kl}-1$ derivatives, that is, are

$$F_{k1}(0),\ F_{k1}(1),\ldots\ldots F_{k1}\,(q_{k1}-1).$$

It will be observed that the number of functions thus obtained from the first l members of a group is equal to the multiplicity of the lth member itself, and that from this lth member alone we can obtain this number of integrals, namely $F_{kl}\ (0),\ F_{kl}\ (1),\ldots F_{kl}\ (q_{kl}-1)$, but examples can be found to shew that the first $q_{k,\,l-1}$ members of this last set of functions are not necessarily linearly independent and indeed that some or all of them may vanish identically. The total number of integrals obtainable is the sum of the multiplicities of the last members of the various groups, and this is equal to or less than the order of the equation according as it is all or only some of the integrals of the equation which are regular.

In conclusion, we must return to the construction of integrals of equation (17) on the basis of an equation of the form (3) involving more than one parameter. There are two possibilities, illustrated by the examples already given, which might be thought distinct: the vanishing of the function $h(p_1, p_2,\ldots p_r)$ may be due solely to the taking of a particular value b_{tk} by one of the parameters p_k, independently of the other parameters, or it may be due to the satisfying of a relation really determining one parameter in terms of the others. But there is nothing vital in this distinction; in either case $r-1$ of the parameters remain arbitrary, and the function $F(x, p_1, p_2,\ldots p_r)$ is a function from which in general not all the arbitrary elements will disappear in virtue of the one equation $h(p_1, p_2,\ldots p_r)=0$. It is natural to regard as the

general case that in which, after substituting in $F(x, p_1, p_2, \ldots p_r)$ for one of the parameters in terms of the others in accordance with the equation $h(p_1, p_2, \ldots pr)=0$, the function remains a function of x and of $r-1$ parameters; then it is a solution of the differential equation (17) involving $r-1$ constants, so that $r-1$ cannot be greater than n, the order of the equation, and if $r-1$ is equal to n, the solution is general. But it is equally conceivable for $F(x, p_1, p_2, \ldots p_r)$ to be of the form

$$H(x)+h(p_1, p_2 \ldots p_r)\, K(x, p_1, p_2, \ldots p_r)$$

in which case equation (3) assists us to only a particular solution of equation (17); any intermediate case is possible, and in fact the real difference between various functions $h(p_1, p_2, \ldots p_r)$ from the point of view of utility in this connection resides in the generality remaining in $F(x, b_{t1}, b_{t2}, \ldots b_{tr})$ where $b_{t1}, b_{t2} \ldots b_{tr}$ is a zero-set of the function $h(p_1, p_2, \ldots p_r)$. And what it is important to observe is that if a zero-set varies continuously it may yield always the same integral of the equation (17), or it may yield any number of integrals which are analytically distinguishable, but it cannot yield more than $r-2$ functions which are linearly independent solutions of the differential equation.

Multiplicity of a zero-set is the last point to which we have to refer. If the function h together with one of its derivatives of order m and all those derivatives of lower orders which are subsidiary to this particular derivative vanishes for the values $b_{t1}, b_{t2} \ldots b_{tr}$ of the parameters, then the function F and all its corresponding derivatives including that of the mth order, assume for this set of values of the parameters forms which are integrals of equation (17).

Again we may illustrate from equation (10) how the fact of having more than one parameter may be utilised. To obtain more than one solution of the equation

$$(D^2-2D+2)y=0$$

from the one-parameter results (14), (15), (16) we have to introduce complex numbers, and then the factors of $F(x, p)$ exchange parts in a curious fashion. The roots of $h(p)$ are $1, i, -i$, all simple, the function $F(x, 1)$ is real, but the functions $F(x, i)$, $F(x, -i)$ are complex, and

$$(1+i)\, F(x, 1)+iF(x, i)+F(x, -i)=0.$$

If we take as our basis the two-parameter results (11), (12), (13) we have a zero-set of $h(p, q)$ which varies continuously, and $-2(p-1)\, qe^{(p+iq)x}$ is an integral for all values of p, q satisfying the equation $h(p,q)=0$. The only pairs of real values of p, q such that $h(p, q)=0$ are $p=1$, $q=\pm 1$; these values cause the integral just found to vanish identically, so that a certain trivial amount of justification has to be

made of the statement that the finite functions $e^{(1+i)x}$, $e^{(1-i)x}$ are in fact integrals, and even then the functions are complex. Trouble is avoided by remarking that the sets (1, 1) and (1, −1) are double zero-sets of $h(p, q)$ with respect to both the parameters p, q, so that four particular integrals (a number equal to the degree of $h(p, q)$ in the two parameters) are given as the values of $\partial F/\partial p$ and $\partial F/\partial q$ for the two sets. The two values of $\partial F/\partial p$ are $\mp 2e^x \cos x$ and are not linearly independent, and the two values of $\partial F/\partial q$ are $\mp 2e^x \sin x$, and these though not linearly independent of each other are linearly independent of those already found. The equation being of order two, its complete solution is now in our hands, but we may notice that the values of $\partial^2 F/\partial p\, \partial q$ for the two zero-sets are also particular integrals; as a matter of fact they coincide, both of them being $-2e^x \cos x$.

Some Infinite Determinants.

BY S. NARAYANA AIYAR, M. A.

Lemma.

If

$$\begin{vmatrix} a_1 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ a_2 & a_1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ a_3 & a_2 & a_1 & 1 & 0 & \cdots & 0 & 0 \\ a_4 & a_3 & a_2 & a_1 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n-1} & a_{n-2} & \cdots & \cdots & \cdots & \cdots & a_1 & 1 \\ a_n & a_{n-1} & \cdots & \cdots & \cdots & \cdots & a_2 & a_1 \end{vmatrix} = (-1)^n A_n$$

then

$$\begin{vmatrix} A_1 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ A_2 & A_1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ A_3 & A_2 & A_1 & 1 & 0 & \cdots & 0 & 0 \\ A_4 & A_3 & A_2 & A_1 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ A_{n-1} & A_{n-2} & \cdots & \cdots & \cdots & \cdots & A_1 & 1 \\ A_n & A_{n-1} & \cdots & \cdots & \cdots & \cdots & A_2 & A_1 \end{vmatrix} = (-1)^n a_n. \quad \ldots \quad (1).$$

Demonstration. By giving to n successive integral values from unity upwards and by developing the determinants thus obtained from the first determinant, we have the relations

$$A_1 + a_1 = 0$$

$$A_2 + A_1 a_1 + a_2 = 0.$$

$$A_3 + A_2 a_1 + A_1 a_2 + a_3 = 0.$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots$$

$$A_n + A_{n-1} . a_1 + A_{n-2} \ a_2 + \ldots A_2 . \ a_{n-2} + A_1 . \ a_{n-1} + a_n = 0. \quad \ldots \quad (2)$$

These relations are symmetrical in A and a. If the second determinant should be true the same relations should hold. Hence the *Lemma*.

Other demonstrations may be given but the above seems to be the simplest.

Prop. I. The infinite determinant

$$\begin{vmatrix} 1 & x & x^2 & x^3 & x^4 & x^5 & \dots & \text{to } \infty \\ a_1 & 1 & 0 & 0 & 0 & 0 & \dots & \dots \\ a_2 & a_1 & 1 & 0 & 0 & 0 & \dots & \dots \\ a_3 & a_2 & a_1 & 1 & 0 & 0 & \dots & \dots \\ a_4 & a_3 & a_2 & a_1 & 1 & 0 & \dots & \dots \\ a_5 & a_4 & a_3 & a_2 & a_1 & 1 & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \text{to } \infty & \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{vmatrix}$$

is equal to

$$\frac{1}{1+a_1 x+a_2 x^2+a_3 x^3+a_4 x^4+a_5 x^5+\dots\dots} \qquad \dots (3)$$

Demonstration. It is easily seen that the minor of x^n in the above determinant is $(-1)^n A_n$. Hence expanding the determinant in terms of the constituents of the first row we have it equal to

$$1+A_1 x+A_2 x^2+A_3 x^3+A_4 x^4+\dots \qquad \dots \qquad \dots (a)$$

Taking the same determinant with capital A's substituted for the small a's, the resulting determinant will, by the *Lemma* above, be equal to

$$1+a_1 x+a_2 x^2+a_3 x^3+a_4 x^4+\dots\dots\dots \qquad \dots \qquad \dots (b)$$

Multiplying the two expressions (a) and (b) and arranging the result in ascending powers of x we find, by the results (2) given in the above *Lemma*, the co-efficient of every power of x is zero. Hence the two expressions $1+A_1 x+A_2 x^2+A_3 x^3+\dots\dots$and $1+a_1 x+a_2 x^2+a_3 x^3+\dots$ should be reciprocals when x is less than unity.

Hence the proposition.

Example 1.

$$\begin{vmatrix} 1 & x & x^2 & x^3 & x^4 & \dots \\ \alpha & 1 & 0 & 0 & 0 & \dots \\ \beta & \alpha & 1 & 0 & 0 & \dots \\ 0 & \beta & \alpha & 1 & 0 & \dots \\ 0 & 0 & \beta & \alpha & 1 & \dots \\ 0 & 0 & 0 & \beta & \alpha & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{vmatrix} = \frac{1}{1+\alpha x+\beta x^2}.$$

If a and b are the roots of the equation $1+\alpha x+\beta x^2=0$, expressing $\frac{1}{1+\alpha x+\beta x^2}$ as equal to $\frac{1}{x(a-b)}\left\{\frac{1}{1-ax}-\frac{1}{1-bx}\right\}$

and expanding separately each term in powers of x the co-efficient of x^n is $\frac{a^{n+1}-b^{n+1}}{a-b}$. The minor of x^n in the above determinant is

$$\begin{vmatrix} \alpha & 1 & 0 & 0 & 0 & \cdots \\ \beta & \alpha & 1 & 0 & 0 & \cdots \\ 0 & \beta & \alpha & 1 & 0 & \cdots \\ 0 & 0 & \beta & \alpha & 1 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix} \qquad (4)$$

a determinant of the nth order.

Hence the value of this determinant of the nth order is $\frac{a^{n+1}-b^{n+1}}{a-b}$ where a and b are the roots of the equation $1+\alpha x+\beta x^2=0$.

The same value in another form is given in Burnside and Panton, Vol. II, page 63.

Ex. 2. Consider the determinant

$$\begin{vmatrix} -2\cos\theta & 1 & 0 & 0 & 0 & \cdots \\ 1 & -2\cos\theta & 1 & 0 & 0 & \cdots \\ 0 & 1 & -2\cos\theta & 1 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix} \qquad (5)$$

This is of the nth order and is the co-efficient of x^n in the expansion of $\frac{1}{1-2x\cos\theta+x^2}$ in powers of x and is therefore equal to $\frac{\sin(n+1)\theta}{\sin\theta}$.

Ex. 3. Let $1+p_1x+p_2x^2+p_3x^3\ldots+p_nx^n$ be equal to

$$(1-x\alpha)(1-x\beta)(1-x\gamma)\ldots\ldots(1-x\nu),$$

then the symmetric function

$$\sum\frac{\alpha^{n+r-1}}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)\ldots(\alpha-\nu)}$$

is equal to the determinant of the rth order

$$\begin{vmatrix} p_1 & 1 & 0 & 0 & \cdots & 0 \\ p_2 & p_1 & 1 & 0 & \cdots & 0 \\ p_3 & p_2 & p_1 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ p_r & p_{r-1} & \cdots & \cdots & \cdots & p_1 \end{vmatrix} \qquad (6)$$

For, expressing $\frac{1}{1+p_1x+p_2x^2+\ldots p_nx^n}$ as the sum of n partial fractions and equating the co-efficient of x^n in the expansion of these fractions

with the minor of x^n in the corresponding determinant, we get the above result.

Ex. 4.
$$e^{-x}=\frac{1}{1+x+\frac{x^2}{1\cdot 2}+\frac{x^3}{1\cdot 2\cdot 3}+\frac{x^4}{1\cdot 2\cdot 3\cdot 4}+\dots}$$

and is therefore equal to the infinite determinant

$$\begin{vmatrix} 1 & x & x^2 & x^3 & x^4 & \dots \\ 1 & 1 & 0 & 0 & 0 & \dots \\ \frac{1}{2!} & 1 & 1 & 0 & 0 & \dots \\ \frac{1}{3!} & \frac{1}{2!} & 1 & 1 & 0 & \dots \\ \frac{1}{4!} & \frac{1}{3!} & \frac{1}{2!} & 1 & 1 & \dots \end{vmatrix} \qquad (7)$$

These examples can be indefinitely multiplied by giving particular values to a_1, a_2.........

Prop. II. The fraction

$$\frac{1+b_1x+b_2x^2+b_3x^3+\dots\dots\dots \text{to } \infty}{1+a_1x+a_2x^2+a_3x^3+\dots\dots\dots \text{to } \infty}$$

is equal to the infinite determinant

$$\begin{vmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \text{to } \infty \\ 0 & 1 & x & x^2 & x^3 & x^4 & x^5 & x^6 & \dots \\ b_1 & a_1 & 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ b_2 & a_2 & a_1 & 1 & 0 & 0 & 0 & 0 & \dots \\ b_3 & a_3 & a_2 & a_1 & 1 & 0 & 0 & 0 & \dots \\ b_4 & a_4 & a_3 & a_2 & a_1 & 1 & 0 & 0 & \dots \\ \text{to } \infty & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{vmatrix} \qquad (8)$$

Demonstration. By expanding the determinant in terms of the constituents of the first column we get the determinant given in *Prop. I* multiplied by $1+b_1x+b_2x^2+b_3x^3+\dots$ Hence the above is equal to the fraction

$$\frac{1+b_1x+b_2x^2+b_3x^3+b_4x^4+\dots}{1+a_1x+a_2x^2+a_3x^3+\dots\dots\dots}$$

Cor. By putting $a_1=a_2=a_3=\dots\dots=0$, we have

$$1+b_1x+b_2x^2+b_3x^3+b_4x^4+\dots \text{to } \infty$$

equal to the infinite determinant

$$\begin{vmatrix} 1 & 1 & 0 & 0 & 0 & \text{to } \infty \\ 0 & 1 & x & x^2 & x^3 & \cdots \\ b_1 & 0 & 1 & 0 & 0 & \cdots \\ b_2 & 0 & 0 & 1 & 0 & \cdots \\ \text{to } \infty & & \cdots & \cdots & & \cdots \end{vmatrix} \qquad (9).$$

Example 5. The determinant given in *Prop. II* is equal to the infinite determinant

$$\begin{vmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & x & x^2 & x^3 & x^4 & x^5 & \cdots \\ A_1 & B_1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ A_2 & B_2 & B_1 & 1 & 0 & 0 & 0 & \cdots \\ A_3 & B_3 & B_2 & B_1 & 1 & 0 & 0 & \cdots \\ A_4 & B_4 & B_3 & B_2 & B_1 & 1 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix} \qquad (10)$$

where

$$\begin{vmatrix} a_1 & 1 & 0 & 0 & \cdots & 0 \\ a_2 & a_1 & 1 & 0 & \cdots & 0 \\ a_3 & a_2 & a_1 & 1 & \cdots & 0 \\ \cdots & \cdots & & \cdots & \cdots & \cdots \\ a_r & a_{r-1} & & \cdots & \cdots & a_1 \end{vmatrix} = (-1)^r A_r$$

and

$$\begin{vmatrix} b_1 & 1 & 0 & \cdots & 0 \\ b_2 & b_1 & 1 & \cdots & 0 \\ b_3 & b_2 & b_1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ b_r & b_{r-1} & \cdots & \cdots & b_1 \end{vmatrix} = (-1)^r B_r.$$

This can be proved from the relation

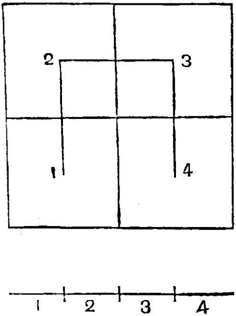
$$\frac{1}{1 + a_1 x + a_2 x^2 + \ldots} = 1 + A_1 x + A_2 x^2 + A_3 x^3 + \ldots$$

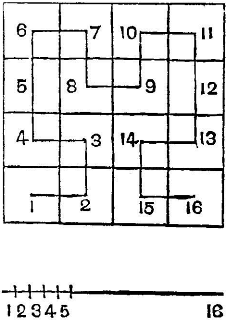
SHORT NOTES.

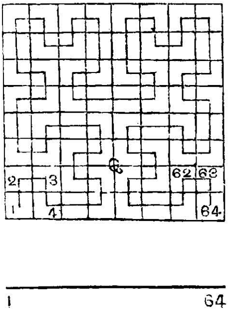
Peano's 'Space filling Curves.'

The notion of a curve, though it may appear simple, is in reality very complex, and often, when the co-ordinates of any point on it are given as functions of a parameter t, a rigorous analytical reasoning leads to results not in conformity with geometric intuition. Peano, an Italian Mathematician, was the first to draw the attention of mathematicians to an extremely curious circumstance; *viz.*, that it is possible to form continuous functions $f(t)$ and $\phi(t)$ such that by varying the parameter t, the point (x, y), given by $x=f(t)$, $y=\phi(t)$, may be made to coincide with all the points inside a square. We thus see that we can obtain, in a way, a 'curve' which fills entirely the area of a square.

The following, which is a modification given by Hilbert of Peano's construction, gives an extremely simple example of a curve possessing the preceding property.







Take a segment of a straight line of unit length and a square of side unity. Then divide the segment of the straight line into four equal parts and the square into four equal squares ; and number the segment of the straight line and the partial squares as in fig. (1). Divide similarly each of the segments of the straight line into four equal parts and each of the partial squares into four others; and number the sixteen squares in the order indicated in fig. (2) so that adjacent segments on the straight line correspond to adjacent squares and each square has a side in common with the succeeding square. Continuing this division of each of the segments of the straight line and each of the partial squares into four equal parts, we obtain 64 squares numbered as in fig. (3) so that as before adjacent segments of the straight line correspond to adjacent squares and each square has a side in common with the succeeding square. We thus see that to each segment upon the straight line (0,1) corresponds a definite square.

We may continue indefinitely this operation. It is then easy to see how to each point of the straight line (0,1) corresponds a point of the initial square. Let p be a point of the straight line. If it is not a point of division, then in the different stages of sub-division there is always a segment containing the point p ; and all these different segments, one from each stage, lie one inside the other and tend towards the point p; to each of the segments corresponds a square and all these squares also lie one within the other ; they diminish indefinitely and tend towards a limiting point, say P, which we take as the point of the square corresponding to the point p. If on the other hand p is a point of division, it determines two series of segments one lying on its right and the other on its left and each having p as the common end-point; corresponding to these two series of segments we have two series of squares, but since the square in one series at any stage is always adjacent to the square in the other series at the same stage, both series define the same point P, which is the common boundary point. Thus again we have corresponding to p a single point P.

Next this correspondence is also continuous. For an interval of length not greater than d (where $4^{-(k+1)} \leq d < 4^{-k}$) with the point p as centre will overlap, at most, into two adjacent intervals at the kth stage ; therefore two, at most, of the corresponding squares and these adjacent to one another will suffice to contain a correspondent for each interval point of the interval d ; and since the diagonal length of two such squares is at most $\frac{\sqrt{5}}{2^k}$, a circle C with centre P and radius

$r > \frac{\sqrt{5}}{2^k}$, will certainly enclose a correspondent for each interval point of d. Now whatever circle C be described having P as centre, we can always determine k so that $\frac{\sqrt{5}}{2^k}$ is less than its radius, and if we take the length of d to lie between $4^{-(k+1)}$ and 4^{-k}, it follows that the correspondent of each and every internal point of d is inside the circle C. Therefore the correspondence is a continuous one.

Also this correspondence is such that corresponding to a point p there is one and only one point P. We may therefore say that, if (x, y) denote the co-ordinates of P, we have $x=f(t), y=\phi(t)$ $(o \leq t \leq 1)$, t being the distance of p from the origin, where f and ϕ are continuous single-valued functions of t. We thus have a '*continuous line*' filling up completely a unit square.

It should, however, be noted that this is not a case of *one-to-one* correspondence. For, though for every point p of the straight line (0,1) we have one and only one point P of the square, yet corresponding to a point P of the square, we may have one, two, or four points of the straight line. If P be a point of the square which does not lie on a dividing line at any stage, then there is one and only one point p of the straight line corresponding to it. If P lies on a dividing line but not at a corner of a square, it will have two different correspondents on the straight line one for each series of squares lying one inside the other and having P as common boundary point. Lastly, if P is at a corner of a square where the dividing lines intersect as the point q in fig (3), it will have four distinct correspondents.

The functions $f(t)$ and $\phi(t)$ thus obtained being continuous functions of t, they also offer a very simple example of *continuous functions not having derivatives*. For, if these functions have derivatives, there will be an arc for the curve considered which is clearly absurd since the curve fills the whole square.

Now a question may be raised whether such a line as is described above can be called a 'curve.' The answer to this will depend on the definition of a *curve*. Jordan originally defined a curve as a 'continuous line" *i.e.*, as "a plane set of points which can be brought into continuous correspondence with the points of a closed segment of a straight line, and raised a query whether it was possible for such a curve to fill up completely a space. In answer to this query Peano gave a construction similar to the above and showed that a curve thus defined may fill up entirely a space; and hence the expression "space filling curve"

has crept into mathematical literature. As this curious circumstance pointed out by Peano does violence to our common concept of a curve, a further restriction has since been introduced into the definition of a curve, *viz.*, that the correspondence should be *one-to-one*; *i.e.*, in analytical definition by $x=f(t)$ and $y=\phi(t)$, f and ϕ must be continuous single-valued functions of t, within given limits, which *do not assume the same pair of values for any two different values of t* within the given limits.

For further information, the reader is referred to Young's *Theory of Sets of Points* and Jordan's *Cours d' Analyse.*

P. V. Seshu Aiyar.

The Generalized Cassinian.

1. The locus of a point which moves in such a way that the product of tangents from it to two fixed circles is constant may be called the *generalized* Cassinian.

2. *The equation of the curve.*

Let the fixed circles be

$$S \equiv x^2+y^2+2gx+k=0,$$
$$S' \equiv x^2+y^2+2g'x+k=0,$$

referred to the line of centres and the radical axis as axes of reference. Then, if (x, y) be any point P on the locus, the tangents t, t' to S, S' are such that

$$t^2=S,\ t'^2=S', \text{ and } tt'=\text{a constant, } b^2 \text{ say.}$$

Hence, the equation of the locus is

$$S.S'=b^4;$$

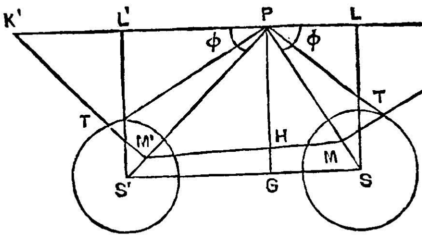
i.e., $\qquad (r^2+2gx+k)\,(r^2+2g'x+k)=b^4. \qquad \ldots \qquad \ldots \ (1)$

Equation (1) is included among the general class of bicircular quartics and is further capable of being written

$$[r^2+(g+g')x+k]^2-(g-g')^2x^2=b^4$$

or $\qquad [r^2+(g+g')x+k]^2=(g-g')^2x^2+b^4 \qquad \ldots \qquad \ldots \ (2)$

3. *If P be a point on the curve, straight lines drawn through T, T' (the points of contact of tangents from P to S, S') perpendicular to SP, S'P will meet the tangent at P in points equi-distant from P.*



Let the perpendiculars TM, T'M' on SP, S'P' meet the tangent at P in K, K'. Then, we have

$$t^2 = PT^2 = PM.PS = PL.PK$$
$$t'^2 = PT'^2 = PM'.PS' = PL.PK.'$$

Also, since $tt' = b^2$,

$$\frac{1}{t}\cdot\frac{dt}{ds} + \frac{1}{t'}\frac{dt'}{ds'} = 0.$$

and $$rdr = tdt\,;\ r'dr' = t'dt'. \quad \ldots \quad \ldots \quad \ldots \quad (3)$$

Hence $$\frac{rdr}{t^2 ds} + \frac{r'dr'}{t'^2 ds'} = 0\,;$$

i. e., $$\frac{1}{PM}\,\frac{dr}{ds} + \frac{1}{PM'}\,\frac{dr'}{ds'} = 0.$$

But $$dr/ds = \cos\phi, -dr'/ds' = \cos\phi'.$$

∴ $$PM \sec\phi = P'M' \sec\phi',$$

or $$PK = P'K'.$$

Cor. $$r\cos\phi : r'\cos\phi' = t^2 : t'^2. \quad \ldots \quad \ldots \quad \ldots \quad (4)$$

4. *If the normal at P meet SS' in G, and MM' in H, $SG : S'G = t^2 : t'^2$, and $MH : M'H = PM^2 : PM'^2$,*

For,

$$SG : S'G = PL : PL'$$
$$= PL.\ PK : PL.\ PK',\ \text{since}\ PK = PK',$$
$$= t^2 : t'^2 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (5)$$

Also,

$$MH : M'H = PM\cos\phi : PM'\cos\phi'$$
$$= PM^2 : PM'^2,\ \text{since}\ \cos\phi/\cos\phi' = PM/PM' \quad \ldots \quad (6)$$

Cor. The normal is the symmedian through P of the triangle MPM'.

This follows from the well-known property of the symmedian of a triangle.

5. *Radius of Curvature at P.*

Let PSS$'$=θ, PS$'$S=θ', PGS=ψ; then

$$\psi-\theta'=\tfrac{1}{2}\pi-\phi'\ ;\ \psi+\theta+\tfrac{1}{2}\pi-\phi=\pi$$

$$\therefore \qquad \frac{d\psi}{ds}-\frac{d\theta'}{ds}+\frac{d\phi'}{ds}=0\ ;\ \frac{d\psi}{ds}+\frac{d\theta}{ds}-\frac{d\phi}{ds}=0. \qquad \dots (7)$$

Now, differentiating the relation (4)

$$r\cos\phi : r'\cos\phi'=t^2 : t'^2,$$

we have

$$\frac{1}{r}\frac{dr}{ds}-\tan\phi\frac{d\phi}{ds}-\frac{2}{t}\cdot\frac{dt}{ds}=\frac{1}{r'}\frac{dr'}{ds}-\tan\phi'\frac{d\phi'}{ds}-\frac{2}{t'}\frac{dt'}{ds};$$

Using (3), (7) and remembering that $rd\theta/ds=-\sin\phi$, $r'd\theta'/ds=\sin\phi'$, this may be written

$$\frac{1}{r}\cos\phi-\tan\phi\left(\frac{1}{\rho}-\frac{\sin\phi}{r}\right)-2\frac{r\cos\phi}{t^2}=-\frac{1}{r'}\cos\phi'-$$
$$\tan\phi'\left(\frac{\sin\phi'}{r'}-\frac{1}{\rho}\right)+\frac{2\,r'\cos\phi'}{t'^2}.$$

Hence, simplifying,

$$\frac{1}{\rho}(\tan\phi+\tan\phi')=\frac{1}{r\cos\phi}+\frac{1}{r'\cos\phi'}-\frac{2r\cos\phi}{t^2}-\frac{2r'\cos\phi'}{t'^2};$$

i. e.,
$$\frac{1}{\rho}(\tan\phi+\tan\phi')=\frac{1}{\text{PL}}+\frac{1}{\text{PL}'}-\frac{4}{\text{PK}}. \qquad \dots \qquad \dots (8)$$

6. The pedal equation with reference to the foot of the radical axis of S, S$'$ as origin, is found as follows :—

By § 2

$$\left.\begin{aligned} t^2&=r^2+2gr\cos\theta+k\\ t'^2&=r^2+2g'r\cos\theta+k\\ tt'&=b^2.\end{aligned}\right\} \qquad \dots \qquad \dots \qquad \dots (9)$$

If p denote the perpendicular from the origin O on the tangent at P,

$$p=r\sin\phi. \qquad \dots \qquad \dots \qquad \dots (10)$$

Also, differentiating the first three equations (9) with respect to r and remembering that $rd\theta/dr=\tan\phi$, we have

$$t\frac{dt}{dr}=r+\frac{g\cos(\theta+\phi)}{\cos\phi};\ t'\frac{dt'}{dr}=r+\frac{g'\cos(\theta+\phi)}{\cos\phi};$$

and
$$\frac{1}{t}\frac{dt}{dr}+\frac{1}{t'}\frac{dt'}{dr}=0.$$

$$\text{Thus } \frac{1}{t^2}\left[r+\frac{g\cos(\theta+\phi)}{\cos\phi}\right]+\frac{1}{t'^2}\left[r+\frac{g'\cos(\theta+\phi)}{\cos\phi}\right]=0. \qquad \dots (11)$$

Eliminating t, t', θ, ϕ from equations (9), (10), (11) we get the pedal equation.

As an example, consider the ordinary oval of Cassini. Here we may put

$$g=c=-g'\ ;\ k=c^2\ ;$$

so that the above equations become

$$t^2=r^2+2cr\cos\theta+c^2,$$
$$t'^2=r^2-2cr\cos\theta+c^2,$$
$$tt'=b^2, \text{ or } r^4+c^4-2c^2r^2\cos 2\theta=b^4,$$
$$p=r\sin\phi,$$
$$r(t^2+t'^2)+c\cos(\theta+\phi)\sec\phi\,(t'^2-t^2)=0.$$

Substitute from the first two in the last and we get

$$2r(r^2+c^2)-4c^2(\cos\theta+\sin\theta\tan\phi)r\cos\theta=0.$$

i.e. $$\sqrt{(r^2-p^2)}\,\{(r^2+c^2)-2c^2\cos^2\theta\}=2c^2p\sin\theta\cos\theta.$$

$\therefore$ $$(r^2-p^2)(r^2-c^2\cos 2\theta)^2=c^4p^2(1-\cos^2 2\theta)$$

or $$p^2(c^4+r^4-2c^2r^2\cos 2\theta)=r^2(r^2-c^2\cos 2\theta)^2$$

or $$p^2.b^4=r^2(r^2-c^2\cos 2\theta)^2$$

or $$p\,b^2=r(r^2-c^2\cos 2\theta).$$

Hence $$2rp.b^2=2r^4-2c^2r^2\cos 2\theta$$
$$=r^4+b^4-c^4\ ;$$

which is the pedal equation of the Cassinian. (*Vide*: Basset, § 251.)

M. T. NARANIENGAR.

Cyclotomic Equations.

1. If N be a prime number and a a number prime to N such that

$$a^e\equiv 1 \pmod N,$$

e being the smallest power of a for which this is possible, then $(N-1)$ is divisible by e.

For, by Fermat's Theorem

$$a^{N-1}\equiv 1 \pmod N.$$

$\therefore$ (a^e-1) must be a factor of $a^{N-1}-1$.

i.e. $(N-1)$ must be divisible by e.

In other words, we may put $(N-1)$ equal to $e.f$.

2. Again, let b be a primitive congruence root of N, so that $(N-1)$ is the lowest power of b for which the congruence

$$b^{N-1}\equiv 1 \pmod N$$

is possible.

Take the terms of the series

$$\left.\begin{array}{ccccc} 1, & a, & a^2, & \dots & a^{e-1} \\ b, & ba, & ba^2, & \dots & ba^{e-1} \\ b^2, & b^2a, & b^2a^2, & \dots & b^2a^{e-1} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ b^{f-1}, & b^{f-1}a, & b^{f-1}a^2, & \dots & b^{f-1}a^{e-1}. \end{array}\right\} \qquad (1)$$

(i) *No two terms of this set can be congruent to N*:
For if possible let

$$a^pb^q \equiv a^{p'}b^{q'}, \text{ where } q'>q.$$

Then $\qquad a^{p-p'} \equiv b^{q'-q}$, if $p>p'$;

otherwise $\qquad a^{e+p-p'} \equiv b^{q'-q}$, since $a^e \equiv 1$.

In other words, $a^{e'} \equiv b^{f'}$, where $e'<e$ and $f'<f$.

$\therefore \qquad a^{ee'} \equiv b^{ef'}$, or $b^{ef'} \equiv 1$,

which is contrary to the supposition, since $ef'<ef<N-1$.

Hence, no two terms of the above set are congruent with respect to N.

(ii) *Also*, $a^e \equiv bf'$; $b^f \equiv a^{e'}$.

For, the above set contains ef or $(N-1)$ terms, none of which are exactly divisible by N and the remainders are all different. Since there cannot be more than $(N-1)$ different remainders (zero being excluded) when the divisor is N, it follows that a^e and b^f should be congruent to one of the terms included in the above set. They cannot be congruent to $a^p.b^q$ since this would mean that

$$a^{e'} \equiv b^{f'} \, [e'<e, f'<f], \text{ which is impossible as proved in (i).}$$

Hence $\qquad a^e \equiv b^{f'}$, $[f'<f]$

and $\qquad b^f \equiv ae'$. $[e'<e]$

3. *To solve the cyclotomic equation*

$$(x^N-1)/(x-1)=0.$$

Put

$$\left.\begin{array}{l} y_1 = \alpha + \alpha^a + \alpha^{a_2} \dots\dots\dots\dots\dots\dots + \alpha^{a^{e-1}} \\ y_2 = \alpha^b + \alpha^{ba} + \alpha^{ba_2} \dots\dots\dots\dots\dots + \alpha^{ba^{e-1}} \\ y_f = \alpha^{b^{f-1}} + \alpha^{\quad f-1} + \dots\dots\dots + \alpha^{b^{f-1}a^{e-1}} \end{array}\right\}$$

where α is a root of the cyclotomic equation.

It is easy to prove that the y's are the roots of a *rational* equation of the fth. degree. [*Cf*: Cajori: *Theory of Equations*, § 180.]

M. T. NARANIENGAR.

Differential Formulae in Astronomy.

The following differential formulae connecting the small variations of the parts of a spherical triangle are of fundamental importance in Astronomy:

$$\delta a = \cos B.\, \delta c + \cos C.\, \delta b + \sin h_a.\, \delta A \qquad \ldots \qquad \ldots \ (1)$$

$$\delta A = -\cos b\, \delta C - \cos c\, \delta B + \sin h_a.\, \delta a. \qquad \ldots \qquad \ldots \ (2)$$

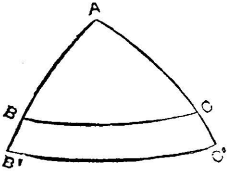
These are easily obtained analytically.

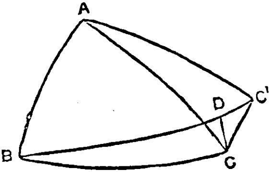
[*Vide*: (1) Simon Newcomb: *Sphl. Astro.*, § 6, and App. I, C.; (2) W. Chavenet: *Astro.*, Vol. I., p. 50.]

The following geometrical proof may be of interest:—

First, let b and c alone vary in the formula

$$\cos a = \cos b \cos c + \sin b \sin c \cos A.$$





Project B′C′ on BC; then

(i) *Increment of* BC $=$ BB′ cos B $+$ CC′ cos C
$= \delta c \cos B + \delta b \cos C.$

Next, let A alone vary.

(ii) *Increment of* BC $=$ CC′ cos CC′B
$=$ CC′ sin AC′B
$= \sin b.\delta A. \sin C$
$= \sin h_a.\delta A,$

in the limit, since AC′C $=90°$ ultimately.

Hence when b, c, A all vary, *the total increment* of BC is

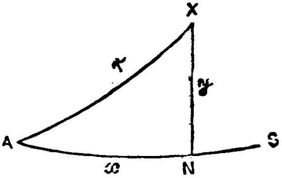
$$\delta a = \delta c \cos B + \delta b \cos C + \sin h_a\, \delta A,$$

which is formula (1).

A similar proof holds with regard to formula (2).

Example: To find the effects of aberration on the latitude and longitude of a star.

The effect of aberration is to give a star X a displacement XX' towards a point A, 90° behind the sun S, such that



$$XX' = k \sin XA,$$

k being the constant of aberration.

Now in the triangle XAN, the angles A and N are unchanged by aberration and therefore by applying formula (1), we have

$$\delta y = \delta r \cos X + \delta x \cos N = \delta r \cos X ; \ldots \qquad \ldots \quad (3)$$

$$\delta r = \delta y \cos X + \delta x \cos A, \quad \ldots \quad \ldots \quad \ldots \quad (4)$$

But $\delta r = -k \sin r, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (5)$

$\therefore \quad \delta y = -k \sin r \cos X = -k \cos x \sin y, \ldots \quad \ldots \quad (6)$

since $\sin A = \cos X \sec x = \sin y \operatorname{cosec} r$.

Again, from (4), (5) and (6)

$$\begin{aligned} \delta x \cos A &= -k \sin r + k \sin r \cos^2 X \\ &= -k \sin r \sin^2 X \\ &= -k \sin x \sin X \\ &= -k \sin x \cos A \sec y, \end{aligned}$$

since $\sin X = \cos A \sec y = \sin x \operatorname{cosec} r$.

$\therefore \quad \delta x = -k \sin x \sec y. \quad \ldots \quad \ldots \quad \ldots \quad (7)$

Remembering that $90° - x = NS$ = difference of longitudes of the sun and the star, we get the usual expressions for the aberration in latitude and longitude from (6) and (7).

V. K. Aravamudan.

The Face of the Sky for May and June 1915.

Sidereal time at 8 p.m.

	May.				June.		
D.	H.	M.	S.		H.	M.	S.
1	10	33	42	... ...	12	35	55
8	11	1	18	... ...	13	3	31
15	11	28	54	... ...	13	31	7
22	11	56	30	... ...	13	58	43
29	12	24	6	... ...	14	26	19

From this table the constellations visible during the evening can be ascertained by a reference to their positions as given in the star-chart.

Phases of the Moon.

		May.					June.		
		D.	H.	M.			D.	H.	M.	
Last Quarter	...	6	10	53 A.M.	...	4	10	2 P.M.
New Moon	...	14	9	1 ,,	...	13	0	27 A.M.
First Quarter	...	22	10	20 ,,	...	20	7	54 P.M.
Full Moon	...	29	3	3 ,,	...	27	9	57 A.M.

The Sun

enters Cancer (Solstice) on 22 June at 5-59 A.M.

The Planets.

Mercury is in superior conjunction with the Sun on May 1. Thereafter it is an evening star attaining its greatest elongation (23° 17′ E) on June 1. It is stationary on June 14 and is in inferior conjunction with the Sun on June 27. It is in conjunction with the Moon on May 15 and on June 14 and with Saturn on May 31.

Venus is a morning star. It is in conjunction with the Moon on May 11 and June 10 and with Mars on May 14.

Mars is in conjunction with the Moon on May 11 and June 9.

Jupiter is in quadrature on June 19. It is in conjunction with the Moon on May 9 and June 6.

Saturn is in conjunction with the Sun on June 28; it is in conjunction with the Moon on May 17 and June 14.

Uranus is in quadrature on May 7 and is stationary on May 21. It is in conjunction with the Moon on May 6, June 2, and June 30.

Neptune is in conjunction with the Moon on May 19 at 8-22 P.M., and June 16.

V. RAMESAM.

SOLUTIONS.

Question 429.

(S. NARAYAN, B.A., L.T.) :—Through any point P in the plane of a triangle ABC, lines are drawn so that the intercepts made by the arms of each of the three angles may be bisected at P. Show that their six points of intersection with the three sides lie on a conic X which will be an ellipse, two parallel lines, or an hyperbola, according as P lies within, on, or without the maximum inscribed ellipse of ABC, and that X will be two intersecting lines through P, if P lies on the radical axis of the circumcircle and the polar circle of ABC.

Solution by N. Sankara Aiyar and 'Zero.'

Let P be the point (f, g, h) ; then the three bisected lines through P are obviously parallel to

$$\frac{\beta}{g}+\frac{\gamma}{h}=0,\ \frac{\gamma}{h}+\frac{\alpha}{f}=0,\ \frac{\alpha}{f}+\frac{\beta}{g}=0,$$

which are the harmonic conjugates of AP, BP, CP. Their equations are therefore of the forms

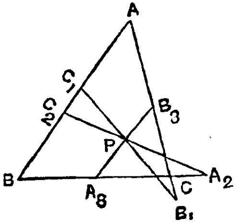
$$\lambda I+\beta/g+\gamma/h=0, \text{ \&c.}$$

where I denotes the lines at infinity, *viz* : $a\alpha+b\beta+c\gamma$. The value of λ is obtained from the condition that the line should pass through P. Thus

$$2\lambda k+2=0. \quad [2k\equiv af+bg+ch]$$

Hence the three bisected lines are

$$I-k\ (\beta/g+\gamma/h)=0, \text{ \&c.}$$



Also, evidently B_3C_2 is parallel to BC and its equation is

$$k\alpha-fI=0.$$

Any conic through the four points $B_1C_1B_3C_2$ is thus of the form

$$(k\alpha-fI)\ \{\ k(\beta h+\gamma g)-gh\ I\ \}\ +\lambda\beta\gamma=0$$

where λ is variable. If $\lambda = fk^2$, the conic becomes

$$k^2\Sigma(f\beta\gamma) - kI\,\Sigma(fg\alpha) + fgh\,I^2 = 0,$$

which by its symmetry must pass through A_2, A_3.

The line $I=0$ meets this where $\Sigma(f\beta\gamma)=0$. Hence the conic will be a hyperbola, parabola, or ellipse according as $\Sigma(af)^{\frac{1}{2}}$ is positive, zero, or negative; that is, according as (f, g, h) lies without, on, or, within the maximum inscribed ellipse $\Sigma(a\alpha)^{\frac{1}{2}}=0$.

If $\Sigma(af)^{\frac{1}{2}}=0$, $C_1\,B_3\,A_2$ and $C_2\,A_3\,B_1$ will be collinear. For, since B_1C_1 is $I-k(\beta/g+\gamma/h)=0$, the co-ordinates of C_1 are given by

$$I-k(\beta/g+\gamma/h)=0, \text{ and } \gamma=0.$$

i.e.,

$$a\alpha+(b-k/g)\,\beta=0, \text{ and } \gamma=0.$$

Similarly B_3, A_2 are given by

$$c\gamma+(a-k/f)\alpha=0,\ \beta=0;$$

and

$$b\beta+(c-k/h)\gamma=0,\ \alpha=0.$$

Hence, the condition of collinearity of $C_1\,B_3\,A_2$ is

$$abc=(a-k/f)(b-k/g)(c-k/h);$$

which on simplification reduces to

$$k^3-k^2(af+bg+ch)+k(abfg+bcgh+cahf)=0;$$

or

$$(af+bg+ch)^2=4(ab\,fg+\ldots\ldots),$$

or

$$\Sigma(af)^{\frac{1}{2}}=0,$$

since $2k=af+bg+ch$.

Question 438.

(A. C. L. Wilkinson, M.A., F.R.A.S.):—If through a point P in the plane of a triangle ABC, straight lines are drawn parallel to the sides, show that the six points of intersection with the sides lie on a conic which is an ellipse, parabola, or hyperbola according as P lies within, on, or outside the maximum inscribed ellipse of ABC. Shew also that it is two straight lines if P lies on the minimum ellipse circumscribing ABC. (Suggested by Q. 429.)

Solution by N. Sankara Aiyar and Zero.

The equations of the lines through P parallel to sides of ABC are evidently

$$fI-2k\alpha=0, \textit{ etc.}$$

The six points of intersection of these with the sides therefore lie on

$$(fI-2k\alpha)(gI-2k\beta)(hI-2k\gamma)=-8k^3\,\alpha\beta\gamma,$$

which reduces to the conic

$$fgh\, I^2 - 2k\, I\, \Sigma(\alpha gh) + 4k^2 \Sigma(f\beta\gamma) = 0.$$

As in Q. 429, this is a hyperbola, parabola or ellipse according as P is without, on, or within the maximum inscribed ellipse.

The condition for two straight lines is obtained directly as in Question 429, and is found to be

$$(af-2k)\,(bg-2k)\,(ch-2k) = abcfgh\,;$$

or
$$4k^2 - 2k\,\Sigma(af) + \Sigma(abfg) = 0\,;$$

i.e.
$$\Sigma(ab\,fg) = 0,$$

which is identical with the condition that the point P (f, g, h) should lie on the minimum circumscribed ellipse of the triangle ABC.

Question 480.

(V. Ramaswami Aiyar) :—If a straight line cut a three-cusped hypocycloid at P, Q, R, S, show that the extremities of the tangent chords at P, Q, R, S, lie on a rectangular hyperbola.

If S=0, S′=0 be two quartics then S+k S′=0 is the equation of the family of quartics passing through the sixteen points of intersection of S=0 and S′=0.

Let S=0 denote the tricusp $\alpha^{-\frac{1}{2}} + \beta^{-\frac{1}{2}} + \gamma^{-\frac{1}{2}} = 0$; and S′=0 denote the four tangent chords, *viz.*, $\alpha x_1^3 + \beta y_1^3 + \gamma z_1^3 = 0$, etc., such that $(x_1\, y_1\, z_1)$ etc., lie on the given line $l\alpha + m\beta + n\gamma = 0$.

Hence we have to prove that S+k S′=0 may for a suitable value of k be made to represent the line $l\alpha + m\beta + n\gamma = 0$ twice, and a conic on which lie the eight points of intersection of the tangent chords with the tricusp.

In other words

$$\Sigma\alpha^2\beta^2 - 2\alpha\beta\gamma\Sigma(\alpha) + k\Pi(\alpha x_1^3 + \beta y_1^3 + \gamma z_1^3)$$
$$\equiv (l\alpha + m\beta + n\gamma)^2 \{\, \Sigma(u\alpha)^2 + 2\Sigma(u'\beta\gamma) \,\}.$$

Equating coefficients of like terms, we get

$$ul^2 = kx_1^3 x_2^3 x_3^3 x_4^3,\quad vm^2 = ky_1 y_2^3 y_3^3 y_4^3;\ \text{etc.} \qquad \ldots (1)$$
$$2ulm + 2u'l, = k\Sigma(x_1^3 y_2^3 x_3^3 x_4^3)\ ;\ \text{etc.} \qquad \ldots (2)$$

But $x_1x_2x_3x_4 : y_1y_2y_3y_4 : z_1z_2z_3z_4 = l : m : n.$ (J.I.M.S., Vol. V, p. 84)

$$\therefore\ \frac{ul^2}{l^3} = \frac{vm^2}{m^3} = \frac{wn^2}{n^3}\ ;\ i.e.,\ u : v : w = l : m : n.$$

Again
$$\Sigma\{\, x_1^3 y_2^3 x_3^3 x_4^3 \,\} = x_1^3 x_2^3 x_3^3 x_4^3 \Sigma\left(\frac{y_2^3}{x_2^3}\right),$$

and
$$\Sigma\frac{y_2^3}{x_2^3} = \left(\Sigma\frac{y_2}{x_2}\right)^3 - 3\Sigma\frac{y_2}{x_2}.\Sigma\frac{y_2y_3}{x_2x_3} + 3\Sigma\frac{y_2y_3y_4}{x_2x_3x}.$$

The equation giving $\frac{y_2}{x_2}$ is, (J. I. M. S., Vol. V, p. 84),

$$ly^4+2lxy^3+x^2y^2(l+m+n)+2mx^3y+mx^4=0.$$

$$\therefore \quad \Sigma\frac{y_2^3}{x_2^3}=-8+3\cdot2\cdot\frac{l+m+n}{l}-\frac{6m}{l}=\frac{6n}{l}-2.$$

$$\therefore \quad 2ulm+2u'\,l^2=kx_1^3x_2^3x_3^3x_4^3\left(\frac{6n}{l}-2\right)$$

$$=ul^3\left(\frac{6n}{l}-2\right).$$

$$\therefore \quad 2u'l^2=ul\,(6n-2l-2m).$$

$$\therefore \quad u' : u=3n-l-m : l.$$

Hence the required conic may be written in the form

$$\Sigma(l\alpha^2)+2\Sigma(3l-m-n)\,\beta\gamma=0.$$

This is a rectangular hyperbola, in as much as

$$\Sigma(l)-2\Sigma(3l-m-n)\cos A=\Sigma(l)-\Sigma(3l-m-n)=0,$$

since $2\cos A=2\cos B=2\cos C=1$.

Hence the eight points in question lie on a rectangular hyperbola.

Question 483.

(V. V. S. Narayan) :—Construct a triangle ABC, being given the angle A, the side AB and the distance of the middle point of AC from the symmedian point.

Geometrical Solution by the Proposer.

Given a side and an angle adjacent to it, the locus of the Lemoine point is a conic (Neuberg). Hence, since the angle A and AB are given the conic or the locus of K can be drawn. Take AB and draw BAC so that BAC$=\angle$A. From B draw BD $\perp^r$ to AC and let M be its middle point. Through M draw MPK to cut the conic at P and AC at K so that PK is of given length. Set off KC along AC so that KC$=$KA. Join ABC which shall be the required triangle.

Question 523.

(R. N. Apte, M.A., F.R.A.S.) :—Find the value of

$$\iint xyz\left\{1+\left(\frac{dz}{dx}\right)^2+\left(\frac{dz}{dy}\right)^2\right\}dxdy$$

where $z=c\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)^{\frac{1}{2}}$ and the integration extends over the positive quadrant of $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Additional Solution by T. P. Bhaskara Shastri.

Here $\frac{dz}{dx}=-\frac{c^2x}{a^2z}$; $\frac{dz}{dy}=-\frac{c^2y}{b^2z}$.

The given integral U

$$=c\iint xy\left\{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right\}^{-\frac{1}{2}}\left\{1+\frac{c^2-a^2}{a^4}x^2+\frac{c^2-b^2}{b^4}.y^2\right\}dxdy,$$

where the integration extends to all positive values of x and y subject to the condition $\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2<1$.

$$\therefore\quad U=c\iint xy\left\{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right\}^{\frac{1}{2}}dxdy+\frac{c^3}{a^4}\iint x^3y\left\{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right\}^{-\frac{1}{2}}dx\,dy$$
$$+\frac{c^3}{b^4}\iint xy^3\left\{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right\}^{-\frac{1}{2}}dxdy.$$

By Dirichlet's theorem,

$$U=c.\frac{a^2b^2}{4}.\frac{\Gamma(1)\Gamma(1)}{\Gamma(2)}\int_0^1(1-u)^{\frac{1}{2}}udu$$
$$+\frac{c^3}{a^4}.\frac{a^4b^2}{4}\,\frac{\Gamma(2)\Gamma(1)}{\Gamma(3)}\int_0^1(1-u)^{-\frac{1}{2}}u^2du$$
$$+\frac{c^3}{b^4}.\frac{a^2b^4}{4}\,\frac{\Gamma(1)\Gamma(2)}{\Gamma(3)}.\int_0^1(1-u)^{-\frac{1}{2}}u^2du.$$

Now

$$\int_0^1 u(1-u)^{\frac{1}{2}}du=\frac{4}{15}.$$

$$\int_0^1 u^2(1-u)^{-\frac{1}{2}}du=\frac{16}{15}.$$

$$\therefore\quad U=\frac{ca^2b^2}{4}.\ \frac{4}{15}+\frac{16}{15}.\ \frac{c^3b^2}{4}.\ \frac{1}{2}+\frac{16}{15}.\ \frac{c^3a^2}{4}.\ \frac{1}{2}$$

$$=\frac{c}{15}\left[a^2b^2+2c^2(a^2+b^2)\right].$$

Question 532.

(R. Vythynathaswamy) :—If 2A, 2B, 2C are the mutual inclinations of any three lines in space and 2L, 2M, 2N, the angles which any fourth straight line makes with them, shew that the following identical relation holds

$$\begin{vmatrix} 0 & 1 & 1 & 1 & 1 & 4 \\ 1 & 0 & \sin^2 C & \sin^2 B & \sin^2 L & 1 \\ 1 & \sin^2 C & 0 & \sin^2 A & \sin^2 M & 1 \\ 1 & \sin^2 B & \sin^2 A & 0 & \sin^2 N & 1 \\ 1 & \sin^2 L & \sin^2 M & \sin^2 N & 0 & 1 \\ 4 & 1 & 1 & 1 & 1 & 0 \end{vmatrix} = 0.$$

Remarks by the Proposer.

Consider the product of the rectangular arrays

$$\begin{vmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & l_1\sqrt{2} & m_1\sqrt{2} & n_1\sqrt{2} & 1 \\ 1 & l_2\sqrt{2} & m_2\sqrt{2} & n_2\sqrt{2} & 1 \\ 1 & l_3\sqrt{2} & m_3\sqrt{2} & n_3\sqrt{2} & 1 \\ 1 & l_4\sqrt{2} & m_4\sqrt{2} & n_4\sqrt{2} & 1 \\ 1 & 0 & 0 & 0 & 0 \end{vmatrix} \text{ and } \begin{vmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & l_1\sqrt{2} & m_1\sqrt{2} & n_1\sqrt{2} & 1 \\ 1 & l_2\sqrt{2} & m_2\sqrt{2} & n_2\sqrt{2} & 1 \\ 1 & l_3\sqrt{2} & m_3\sqrt{2} & n_3\sqrt{2} & 1 \\ 1 & l_4\sqrt{2} & m_4\sqrt{2} & n_4\sqrt{2} & 1 \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix}$$

where the l's, m's, n's are direction cosines. The product vanishes identically. The product is easily seen to reduce to the given determinant.

Or we may proceed as follows: From a point O draw line parallel to the given lines and measure along them OA, OB, OC, OD each equal to unity; and write down the identical relation between the distances of the five points O, A, B, C, D expressing the distances in terms of the mutual angles. The result will be that stated in the question.

A simpler relation between the mutual inclinations of any four lines in space is that given in Routh's *Analytical Statics*, *viz.*

$$\begin{vmatrix} 1 & \cos(12) & \cos(13) & \cos(14) \\ \cos(21) & 1 & \cos(23) & \cos(24) \\ \cos(31) & \cos(32) & 1 & \cos(34) \\ \cos(41) & \cos(42) & \cos(43) & 1 \end{vmatrix} = 0,$$

which is easily obtained by considering the product of the rectangular array

$$\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \\ l_4 & m_4 & n_4 \end{vmatrix}$$

by itself.

Question 534.

(Zero) :—A given ellipse is projected orthogonally on a given plane. Shew how to obtain the axes of the projection geometrically.

Solution by A. Narasinga Rao and the Proposer.

Construct an ellipse concentric with the given ellipse, having its minor axis parallel to the common section of the two planes, and the ratio of its axes equal to the cosine of the inclination of the planes. Such an ellipse projects into a circle and therefore the common diameters of the two ellipses project into equal diameters of the projection in question. The bisectors of the angles between these equal diameters will give the axes of the projection.

If we take the limiting cases of the concentric ellipse having double contact with the given ellipse internally and externally, we obtain the axes of the projection in position and magnitude. For, the projections of these limiting ellipses will be concentric circles having double contact with the projection in question and the diameters of contact are therefore the axes in magnitude and position.

Question 537.

(K. J. Sanjana, M. A) :—Prove that

$$\frac{1}{2}\frac{x^2}{2}-\frac{1\cdot3}{2\cdot4}\frac{x^4}{4}+\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\frac{x^6}{6}-\ldots\ldots\ldots$$

$$=\frac{1}{4}\log(1+x^2)+\frac{1}{8}\log(1+x_1^2)+\frac{1}{16}\log(1+x_2^2)+\ldots\ldots$$

where $$x_1^2=\frac{x^4}{4(x^2+1)},\ x_2^2=\frac{x_1^4}{4(x_1^2+1)}\ ;\ \ldots\ldots$$

and that

$$\frac{1}{2}\,\frac{x^2}{2}+\frac{1\cdot3}{2\cdot4}\,\frac{x^4}{4}+\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\,\frac{x^6}{6}+\ldots\ldots$$

$$=-\frac{1}{4}\log(1-x^2)-\frac{1}{8}\log(1+x_1^2)-\frac{1}{16}\log(1+x_2^2)-\ldots\ldots$$

where $\quad x_1^2=\dfrac{x^4}{4(1-x^2)},\quad x_2^2=\dfrac{x_1^4}{4(x_1^2+1)},\ \ldots\ldots$

In the first equality x is any real quantity, in the second any real quantity less than 1.

Additional Solution by N. Sankara Aiyar, M. A.

Now

$$(1+x^2)^{-\frac{1}{2}}=1-\frac{1}{2}x^2+\frac{1\cdot3}{2\cdot4}.\,x^4-\frac{1\cdot3\cdot5}{2\cdot4\cdot6}x^6\ldots\ldots$$

$$\therefore\quad \int\frac{(1+x^2)^{-\frac{1}{2}}}{x}dx=\log x-\frac{1}{2}\,\frac{x^2}{2}+\frac{1\cdot3}{2\cdot4}\,\frac{x^4}{4}-\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\,\frac{x^6}{6}+\ldots\ldots$$

Thus the first series is equal to $\log x+c-\displaystyle\int\frac{(1+x^2)^{-\frac{1}{2}}}{x}dx$

$$=\log x+c+\log\frac{1+\sqrt{1+x^2}}{x}=c+\log(1+\sqrt{1+x^2}).$$

But, since the given series vanishes with x, we have

$$c+\log 2=0.$$

Hence the given series is equal to

$$\log(1+\sqrt{1+x^2})-\log 2$$

$$=\frac{1}{2}\log(1+1+x^2+2\sqrt{1+x^2})-\log 2$$

$$=\frac{1}{4}\log(1+x^2)+\frac{1}{2}\log\frac{x^2+2+2\sqrt{1+x^2}}{2\sqrt{1+x^2}}-\frac{1}{2}\log 2$$

$$=\frac{1}{4}\log(1+x^2)+\frac{1}{2}\log\left(1+\frac{x^2+2}{2\sqrt{1+x^2}}\right)-\frac{1}{2}\log 2.$$

Let $\quad \dfrac{x^2+2}{2\sqrt{1+x^2}}=\sqrt{1+x_1^2}.$

$$\therefore\quad \frac{x^4+4x^2+4}{4(1+x^2)}=4(1+x_1^2).$$

$$\therefore\quad x_1^2=\frac{x^4}{4(1+x^2)}$$

Thus the given series is equal to

$\frac{1}{4}\log(1+x^2)+\frac{1}{2}\log(1+\sqrt{1+x_1^2})-\frac{1}{2}\log 2$

$$=\tfrac{1}{4}\log(1+x^2)+\tfrac{1}{8}(1+x_1^2)+\tfrac{1}{4}\log(1+\sqrt{1+x_2^2})-\tfrac{1}{4}\log 2;$$

and the transformation can be continued *ad infinitum.*

Next $(1-x^2)^{-\frac{1}{2}}=1+\frac{1}{2}x^2+\frac{1\cdot3}{2\cdot4}x^4+\frac{1\cdot3\cdot5}{2\cdot4\cdot6}x^6\ldots\ldots\ldots\ldots$

and the second series $=\int\frac{(1-x^2)^{-\frac{1}{2}}}{x}dx+c-\log x.$

$$=\log\frac{1-\sqrt{1-x^2}}{x}+c-\log x.$$

$$=\log\frac{1-\sqrt{1-x^2}}{x^2}+c$$

$$=\log\frac{1}{1+\sqrt{1-x^2}}+c.$$

$$=-\log(1+\sqrt{1-x^2})+\log 2 \text{ [putting } x=0,$$

$$=-\tfrac{1}{2}\log(2-x^2+2\sqrt{1-x^2})+\log 2.$$

$$=-\tfrac{1}{4}\log(1-x^2)-\tfrac{1}{2}\log\frac{2-x^2+2\sqrt{1-x^2}}{2\sqrt{1-x^2}}+\tfrac{1}{2}\log 2.$$

$$=-\tfrac{1}{4}\log(1-x^2)-\tfrac{1}{2}\log\left(1+\frac{2-x^2}{2\sqrt{1-x^2}}\right)+\tfrac{1}{2}\log 2$$

$$=-\tfrac{1}{4}\log(1-x^2)-\tfrac{1}{2}\log(1+\sqrt{1+x_1^2})-\tfrac{1}{2}\log 2, \text{ as before;}$$

and the transformation can be continued *ad infinitum.*

Question 557.

(PROFESSOR SANJANA) :—At a point P of a parabola the normal chord PQ is drawn meeting the axis in G and PN is the ordinate to the axis, prove that PQ. AN $=2$ SP. PG.

If from P two chords PP_1, PP_2 are drawn normal to the curve at P_1, P_2, prove further that

$SP_1.SP_2=AS.SP$, $PP_1.PP_2=PQ.AN$, where S is the focus.

Shew also that ths circle P_1P_2S passes through the circumcentre of PP_1P_2.

Note by V. V. S. Narayan.

"If from any point three normals are drawn to a parabola the circle through their feet passes through the vertex of the parabola." Add to this the following property enunciated by Mr. M. T. Naraniengar in the Mathematical columns of the *Educational Times* :—"If the normal at P meets the curve at Q and the normal at Q meets the axis in G, then shall APGQ be cyclic." We see that in the present case $GPAP_1P_2$ are concyclic. Now, to prove $SP_1\cdot SP_2=AS\cdot SP$, produce P_1S to cut the circle $GPAP_1P_2$ at R. Then $RS=SP_2$, since $P_1SP_2=2P_1RP_2$,

$\therefore$ $SP_1\cdot SP_2=SP_1\cdot SR=AS\cdot SB=AS\cdot SP.$

Similarly all the other parts of the question can be proved.

The properties ennunciated in the question follow also from the following theorem fully worked out on page 263 of Milne's *Weekly Problem Papers*:

If OP, OQ, OR, the three normals passing through O, cut the parabola again in P′, Q′, R′, then

(i) $$OP'.OQ'.OR' = OD^2.OD'^2.OS/ON^2$$

where D, D′, N are points where the $\perp^r$ from O on the axis cuts the curve and the axis ;

(ii) $$\frac{OP.OQ.OR}{OP'.OR'.OR'} = \frac{ON^2}{OD.OD'}.$$

In the present case O is on the curve and consequently, the three normals are PQ_1, PP_1,PP_2 and applying the above two properties the results stated in the question immediately follow. Milne's solution is, however, *analytical*.

Question 563.

(N. P. Pandya) :—Given two quadrilaterals in the plane of the paper, shew how to draw a straight line bisecting them both.

Solution by A Narasinga Rao.

Let ABCD be one of the quadrilaterals. Produce AB and CD to meet at E.

If PQ be a line bisecting ABCD, P and Q lying in AB and CD respectively the envelope of PQ is a hyperbola H_1 having EB, EC for asymptotes.

Also, the points of contact of all tangents like PQ which cut AB. CD internally lie on a certain arc of this hyperbola.

Similarly all lines PQ which cut AD, BC internally and bisect ABCD touch an arc of a hyperbola H_2 having AD, BC for asymptotes.

In the same way, from the second quadrilateral we obtain two hyperbolas h_1, h_2 the tangents to which bisect the quadrilateral.

A common tangent of H and h with its points of contact in prescribed limits is a solution of the problem.

QUESTIONS FOR SOLUTION.

Correction: In Q. 624 *insert at the end*—'where t, s denote tan α, tan β respectively.'

630. (S. NARAYANA AIYAR, M.A.) :—If

$$A_n=\frac{(a-b)(a-bx)(a-bx^2)(a-bx^3)\ldots\ldots\ldots(a-bx^{n-1})}{(1-x)(1-x^2)(1-x^3)(1-x^4)\ \ldots\ldots\ldots\ (1-x^n)}$$

and
$$B_n=\frac{(b-a)(b-ax)(b-ax^2)(b-ax^3)\ldots\ldots\ldots(b-ax^{n-1})}{(1-x)(1-x^2)(1-x^3)(1-x^4)\ \ldots\ldots\ldots\ (1-x^n)},$$

shew that

$$(-1)^n B_n=\begin{vmatrix} A_1 & 1 & 0 & 0 & 0 & \ldots & 0 \\ A_2 & A_1 & 1 & 0 & 0 & \ldots & 0 \\ A_3 & A_2 & A_1 & 1 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ A_{n-1} & A_{n-2} & A_{n-3} & \ldots & \ldots & \ldots & 1 \\ A_n & A_{n-1} & A_{n-2} & \ldots & \ldots & \ldots & A_1 \end{vmatrix}$$

with a similar determinant for $(-1)^n A_n$ in terms of the constituents $B_1, B_2, B_3\ldots\ldots\ldots B_n$.

631. (S. NARAYANA AIYAR, M.A.) :—Shew that if $\phi\,(a,b,c,d)$ denotes

$$1+\frac{x}{1}\cdot\frac{c-d}{1}\cdot\frac{b+x-1}{a+x-1}+\frac{x(x-1)}{1\cdot 2}\cdot\frac{(c-d)(c-d-1)}{d(d+1)}\cdot\frac{(b+x-1)(b+x-2)}{(a+x-1)(a+x-2)}+\ldots$$

then
$$\frac{\phi(a,\ b,\ c,\ d)}{\phi(c,\ d,\ a,\ b)}=\frac{\Gamma(x+b)\Gamma(x+c)\Gamma(a)\Gamma(d)}{\Gamma(x+a)\Gamma(x+d)\Gamma(b)\Gamma(c)}.$$

632. (R. N. APTE, M.A., F.R.A.S.) :—If $r_1,\ r_2,\ r_3\ldots r_n$ be the distances of a point P in the plane of a regular polygon from the vertices, find $\Sigma(r^2_1 r^2_2 r^2_3)$ in terms of n, a the radius of the circumcircle of the polygon and c the distance of P from the centre of the circle, the summation extending over all possible groups of three vertices.

633. (K. APPUKUTTAN ERADY, M.A.) :—A particle is projected from a point on the boundary of a circular pond of radius a with the velocity

$\sqrt{2gc}$. Prove that, if all directions of projection are equally probable, the chance of its falling into the pond is

$$\frac{1}{2}-\frac{1}{\pi}\int_0^{\frac{\pi}{2}} \sqrt{1-\frac{a}{c}\cos\phi} \quad d\phi$$

634. (E. H. NEVILLE, B.A., B.Sc.) :—On the section of a surface by a plane through the normal ON at a point O is taken a point S at a small distance f from O, and the plane parallel to the tangent plane at O and distant $1/\lambda$ from O meets the normal at O and S to the surface in points L and P. Shew that LP^2 is approximately $(f^2/\lambda^2)\{(k_n-\lambda)^2+a_g\}$, where k_n, a_g are the normal curvature and geodesic torsion of the surface along OS ; deduce that if S traces a small contour round O, λ can be chosen so that P traces a curve which is almost a circle with centre L only if the contour traced by S approximates either to a circle with centre O or to an ellipse whose axes are the principal tangents to the surface, and that λ in the first case must be the mean curvature of the surface but in the second case may have either of two values.

635. (S. KRISHNASAWMI AIYANGAR, M.A.) :—If σ_1, σ_2, σ_3 represent KA, KB, KC where K is the symmedian point, and m_1, m_2, m_3 the medians, of a triangle ABC, show that

1. $\Sigma\{a(b^2+c^2)\,\sigma_1\,m_1\}=\frac{1}{2}\,abc\,(a^2+b^2+c^2)+\frac{3abc}{2}\cdot\frac{a^4+b^4+c^4}{a^2+b^2+c^2}$.

2. $\Sigma a(b^2+c^2)\frac{\sigma_1}{m_1}=4abc.$

3. $\Sigma bc\frac{m_1}{\sigma_1}=\frac{3}{2}(a^2+b^2+c^2)$.

636. (S. KRISHNASWAMI AIYANGAR, M.A.) :—If

$$\sin e^x=\sum_0^\infty a_n\frac{x^n}{\underline{|n}},\ \text{and}\ \cos e^x=\sum_0^\infty b_n\frac{x^n}{\underline{|n}},$$

show that $\quad a_n+a_{n+1}\log p+a_{n+2}\frac{(\log p)^2}{2\,!}+\dots\dots\dots$

$$=p-3^n\frac{p^3}{3\,!}+5^n\frac{p^5}{5\,!}-\dots\dots\dots$$

and find $\sin(e^x+e^{-x})$.

637. (D. KRISHNAMURTI) :—Solve

$$x\left(\frac{d^2y}{dx^2}-y\right)=1+\frac{dy}{dx}.$$

638. (S. B. BELEKAR):—If $\lambda=e^{i}\alpha$, show that the point

$$\frac{a(\lambda^2+1)}{2\lambda},\ \frac{b(\lambda^2-1)}{2i\lambda}$$

lies on

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

If λ_1, λ_2, λ_3 be three points as above, the circle passing through these three is

$$x^2+y^2-\frac{(a^2-b^2)}{4aN}\{N^2+M)+(LN+1)\}x-$$

$$\frac{(a^2-b^2)}{4iLN}\{(N^2+M)-(LN+1)\}y$$

$$+\frac{(MN+L)(a^2-b^2)-2(a^2+b^2)N}{4N}=0,$$

where $L\equiv\Sigma\lambda_1$, $M\equiv\Sigma\lambda_1\lambda_2$, $N\equiv\lambda_1\lambda_2\lambda_3$.

639. (V. V. S. NARAYAN) :—The polar circles of the five triangles external to a pentagon, which are formed by producing its sides have a common orthogonal circle.

640. (SELECTED) :—

The Magic Number Cards.

"This trick consists of six cards with certain of the numbers from 1 to 63 printed on them. The magician asks you to think of a number between 1 and 63 and then presents these cards one at a time with the request that you indicate whether your number is on the card. This done he at once declares the number which you thought of."

Explain the trick and point out any easy extensions of the trick.

641. (SELECTED) :—"There cannot be five prime numbers in arithmetical progression unless their common difference be divisible by $2\times3\times5=30$, except when the first term of the progression is 5."

Barlow's *Theory of Numbers*, p. 65.

Illustrate or criticise the above statement.

642. (S. RAMANUJAN) :—Shew that

$$(1)\ \sum_{n=0}^{\infty}\left(1+\frac{1}{3}+\frac{1}{5}+\dots\frac{1}{2n+1}\right)5^{-n}(2n+1)^{-1}$$

$$=\frac{\pi^2}{4\sqrt{5}}-\frac{\sqrt{5}}{24}[\log(2+\sqrt{5})]^2$$

$$(2)\ \sum_{n=0}^{\infty}\left(1+\frac{1}{3}+\frac{1}{5}+\dots\frac{1}{2n+1}\right)9^{-n}(2n+1)^{-1}$$

$$=\frac{\pi^2}{8}-\frac{3}{8}[\log 2]^2.$$

643. (K. V. ANANTHANARAYANA SASTRI, B.A.) :—Prove that

$$\frac{1}{2n+2}+\frac{1}{2}\,\frac{1}{(2n+4)}+\frac{1\cdot3}{2\cdot4}\,\frac{1}{(2n+6)}+\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\,\frac{1}{(2n+8)}+\dots=\frac{2\cdot4\cdot6\dots2n}{3\cdot5\cdot7\cdot\dots(2n+1)}.$$

644. (K. V. ANANTHANARAYANA SASTRI, B.A.) :—Shew that the length of the fourth positive pedal of a loop of the Lemniscate of Bernoulli is given by

$$18a\int_0^1 \frac{x^8\,dx}{\sqrt{(1-x^4)}}.$$

645. (R. P. VYTHYNATHASWAMY) :—Shew that an identical relation must exist between the mutual moments of any seven lines in space and find it in its determinental form.

About how many given lines should the moment of a line be known to determine the latter completely.